Anisotropy effect on two-dimensional cellular-automaton traffic flow with periodic and open boundaries

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Using computer simulations we investigate, in a version of the Biham-Middleton-Levine model with random sequential update on a square lattice, the anisotropy effect of the probabilities of the change of the motion directions of cars, from up to right (p_{ur}) and from right to up (p_{ru}), on the dynamical jamming transition and velocities under periodic boundaries on one hand and the phase diagram under open boundaries on the other hand. However, in the former case, the sharp jamming transition appears only for $p_{ur}=0=p_{ru}=0$ (i.e., when the cars alter their motion directions). In the open boundary conditions, it is found that the first-order line transition between jamming and moving phases is curved. Hence, by increasing the anisotropy, the moving phase region expands as well as the contraction of the jamming and maximal current phases takes place. Moreover, in the anisotropic case, the transition between the jamming phase (or moving phase) and the maximal current phase is of second order while in the isotropic case, and when each car changes its direction of motion at every time step ($p_{ru}=p_{ur}=1$), the transition is of first order. Furthermore, in the maximal current phase, the density profile decays with an exponent $\gamma \approx \frac{1}{4}$.

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I. INTRODUCTION

Transport phenomena in complex systems, in particular models of highway traffic flow, have attracted much attention in recent years. Much of the effort was concentrated on discrete stochastic models of traffic flow, first proposed by Nagel and Schreckenberg [1], and subsequently studied by many other authors using a variety of techniques [2–5]. Since the introduction of the Nagel-Schreckenberg (NS) model [1], cellular automata became a well established method of traffic-flow modeling. Comparatively low computational cost of cellular-automata models made it possible to conduct large-scale real-time simulations of urban traffic in the city of Duisburg [6] and the Dallas–Forth Worth complex [7]. Compared with the fluid dynamical approaches to traffic-flow problems, the CA models are conceptually simpler, and can be readily implemented on computers. These models have the advantages that they can be easily modified to deal with the effects of different kinds of realistic conditions, such as road blocks and hindrances, traffic accident [8], highway junctions [9], vehicle acceleration [10], stochastic delay due to drivers' reactions [5], anisotropy of car distributions in different driving directions [11], faulty traffic lights [12]. Traffic flow is a kind of many body systems of strongly interacting cars. Recent studies reveal physical phenomena such as the dynamical phase transitions and nonlinear waves [13,14]. When the car density increases, the jamming transition occurs and traffic jams appear. The jamming transitions from the freely moving phase to the jamming phase have been studied by microscopic and macroscopic models. The two-dimensional traffic flow is more complex than the one-dimensional case. It has been investigated only by the cellular-automaton models [15–18].

The NS model [1] is a probabilistic CA model for one-dimensional highway traffic. It considered the effects of acceleration and stochastic delay of vehicles. A vehicle can move at most $v_{\max}$ sites in a time step, where $v_{\max}$ is the maximal velocity. Cheybani *et al.* [19] introduced a constraint during the entry of a vehicle, where at site $i=0$ (i.e., out of the system), a vehicle with the velocity $v=v_{\max}$ created with the probability α can immediately move according to the NS rules [1]. Recently [20] we have studied the effect of boundaries on the NS model, in which we have a vehicle that can enter without constraint, with a probability α, in the first site being on the left side of the road if this site is empty, while a vehicle being on the right in the last site can leave the road with a probability β. There is a free flow and jamming phase separated by a line of first-order transitions; this transition occurs at $\alpha<\beta$ for $p\neq 0$ and $\alpha=\beta$ for $p=0$, and the maximal current phase is obtained only for $v_{\max}=1$ (this case coincides with the asymmetric exclusion process model), otherwise it vanishes.

The two-dimensional models have been presented by Biham, Middleton, and Levine (BML) to mimic the traffic flow in the whole city. The BML model [15] is a simple two-dimensional (square lattice) CA model. Each cell of the lattice represents an intersection of an east-bound and a north-bound street. The cells (intersections) can either be empty or occupied by a vehicle moving to the east or to the north. In order to enable movement in two different directions, east-bound vehicles are updated at every odd discrete time step whereas north-bound vehicles are updated at every even time step. The velocity update of the cars is realized following the ASEP rules; a vehicle moves forward by one cell if the cell in front of it is empty, otherwise the vehicle does not move. This is an alternating movement to a traffic light cycle of one time step. The traffic-flow model is given by a three-state CA on the square lattice. Biham, Middleton, and Levine have studied the traffic-flow problem only in the case $\rho_x=\rho_y$

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$=\rho/2$, where ρ_x and ρ_y are, respectively, the density of cars moving to the right and the density of cars moving upwards, and ρ is the total density of cars. They have found that a jamming transition occurs at a critical density $\rho=\rho_c$ with increasing density of cars. The jamming transition separates between the low density (moving) phase and the high density (jamming) phase. Nagatani [16] has investigated the anisotropy effect of the density of cars on BML models; it was shown that the traffic-jam transition occurs at higher density of cars with increasing difference between the density of right-moving cars and the density of up-moving cars. The difference of the densities of cars has an important effect on the jamming transition.

Our aim in this paper is to study, in a version of the Biham-Middleton-Levine model with random sequential update on a square lattice, the effect of the anisotropy of the probabilities of the change of the motion directions of cars, from up to right (p_{ur}) and from right to up (p_{ru}), on the jamming transition and velocities under the periodic boundary conditions on one hand and the phase diagram and density profile behavior under the open boundary conditions on the other hand. Hence, in addition to the open boundaries effect, in the model we study the trend of the motion of each car at a time that step depends on its direction of motion at the previous time; such conditions have not been considered in Ref. [21] in which the authors have introduced, in a two-dimensional traffic-flow model with parallel dynamics update and periodic boundaries, a randomness parameter γ, which allows one to control the trend of the motion of every car at a time step independently on its direction of motion at the previous time. However, in our case, the sharp jamming transition (from freely moving to jamming phases, i.e., from high to low velocity values) appears only for $p_{ur}=0=p_{ru}=0$. Such transition has been obtained by Cuesta *et al.* [21] but for any values of $\gamma\neq 0.5$. In the open boundary conditions, the topology of phase diagram depends strongly on the anisotropy ($|p_{ru}-p_{ur}|$). However, the first-order line transition between jamming and moving phases is curved. Hence, by increasing the anisotropy, the moving phase region expands as well as the contraction of the jamming and maximal current phases takes place. Moreover, the transition between the jamming phase (or moving phase) and the maximal current phase is of second order in the anisotropic case, while in the isotropic case, and when $p_{ru}=p_{ur}=1$, this transition is of first order. The paper is organized as follows: in the following section we define the model, Sec. III is reserved for results and discussions, the conclusion is given in Sec. IV.

II. MODEL

We consider the BML model on a square lattice of $L\times L$ sites with three-state CA, in both periodic and open boundaries cases. Each site (i,j) of the lattice, with $1\leqslant i\leqslant L$ and $1\leqslant j\leqslant L$, contains either a car moving upwards, a car moving to the right, or empty. At the initial configuration, cars are randomly distributed on the sites of the lattice. Hence, at each time step, the cars are randomly selected within sequential dynamics, if the selected car is in right-moving (up-moving) state, it moves to the right (up) unless the adjacent site on its right (upwards) hand side is occupied by another car, which can be either an up or a right driver. If it is blocked by another car it does not move. After that, if the selected car is in the right-moving (up-moving) state, its state is altered into the up-moving (right-moving) state with the probability p_{ru} (p_{ur}). Then, we perform computer simulations of the CA model starting with a set of random initial conditions for the system size $L=10$–500, the density $\rho=0.0$–1.0 of cars. Each run is obtained after 10 000–50 000 time steps. After a transient period that depends on the system size, on the random initial configuration, and the density of car, the system reaches its asymptotic state. In order to compute the average of any parameter u ($\langle u\rangle$), the values of $u(t)$ obtained in the asymptotic state are averaged.

We denote $\tau(i,j)$ the state of the site (i,j). The periodic boundaries case is defined by the following conditions for $1\leqslant i\leqslant L$:

$$\tau(i,0)=\tau(i,L) \text{ and } \tau(i,L+1)=\tau(i,1),$$

and for $1\leqslant j\leqslant L$:

$$\tau(0,j)=\tau(L,j) \text{ and } \tau(L+1,j)=\tau(1,j).$$

For the open boundaries case, at each time step, if the site $(i,1)$ $[1\leqslant i\leqslant L]$ is empty then a right-moving car is injected

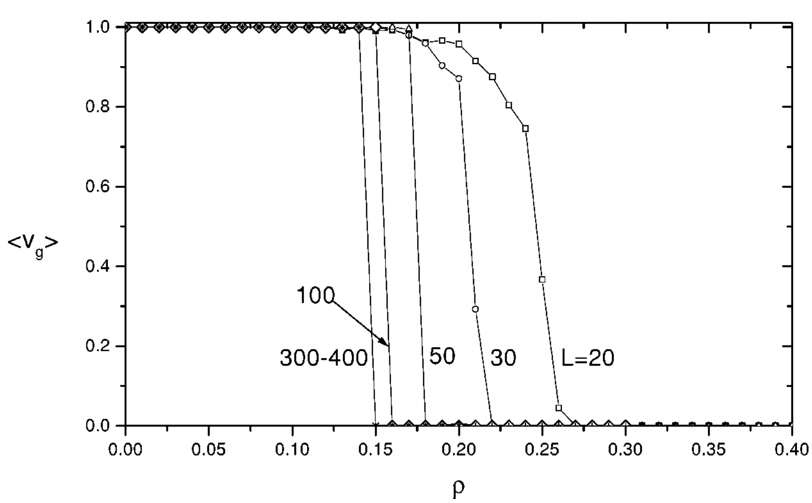

FIG. 1. The variation of the global mean velocity $\langle v_g\rangle$ versus the density for different values of the lattice size L in the case of periodic boundaries.

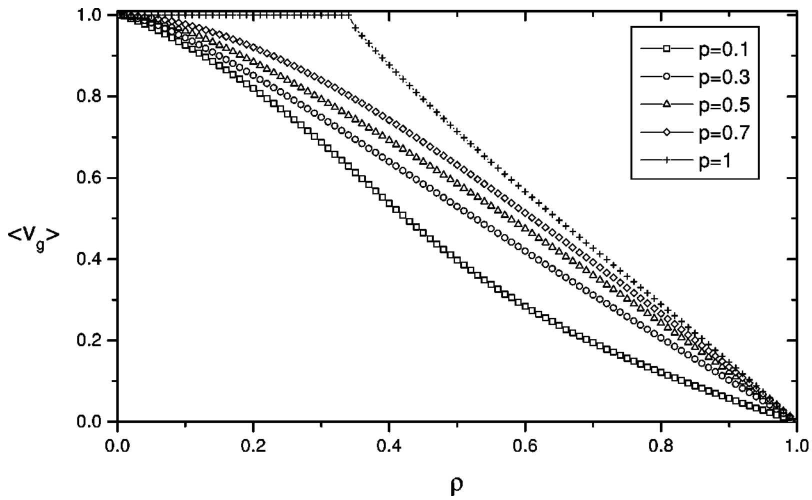

FIG. 2. The variation, in the periodic boundaries case, of the global mean velocity $\langle v_g\rangle$ versus the density for several values of $p=p_{ur}=p_{ru}$ with $L=100$.

with the probability α, while an up-moving car is injected with the probability α if the site $(1,j)$ $[1\leqslant j\leqslant L]$ is empty. However, if an up-moving [right-moving] car reaches one of the sites located in the upper [right] part of the lattice, i.e., the sites (L,j) $[(i,L)]$ with $1\leqslant j\leqslant L$ $[1\leqslant i\leqslant L]$, it leaves the lattice with the probability β.

In the following section, we have used the quantities v_u, v_r, and v_g defined, respectively, as the mean up velocity, the mean right velocity, and the mean velocity. In the periodic boundaries condition, v_u (v_r) is the number of moves performed by the up-moving (right-moving) cars calculated in each time step averaged over the up-moving (right-moving) cars. The same procedure is carried out in order to compute v_g except that we average over all cars. In the open boundaries condition, the quantities are calculated inside the square of length $2l+1$ ($l=6$) centered in the middle of the lattice [i.e., $((L+1)/2,(L+1)/2)$]. We define the quantities d_u, d_r, and d (j_u, j_r, and j) as the density (current) at the middle of the lattice of, respectively, the up-moving cars, right-moving cars, and all types of cars. The currents j_u and j_r at the site (i,j) are defined, respectively, by $\langle u(i,j)[1-g(i+1,j)]\rangle_l$ and $\langle r(i,j)[1-g(i,j+1)]\rangle_l$, where $\langle\ \rangle_l$ is the average over the square $(2l+1)^2$, while the global current is the summation of both the currents, $j=j_u+j_r$. The parameters $u(i,j)$, $r(i,j)$, and $g(i,j)$ are defined as the probability to find the site (i,j) occupied, respectively, by the up-moving car, right-moving car, and any type of cars. Hereafter, we use the following quantities, $\rho_u=\langle d_u\rangle$, $\rho_r=\langle d_r\rangle$, and $\rho=\langle d\rangle$.

III. SIMULATIONS AND RESULTS

A. Periodic boundaries

The variation of the global mean velocity as a function of the density, for $p=p_{ur}=p_{ru}=0$, is given in Fig. 1 for different system sizes. It is clear that the system exhibits two different asymptotic states separated by a sharp jamming transition (discontinuity of the average velocity at the transition between the free moving and the jamming phases). Such transition has been obtained by Cuesta *et al.* [21] but for any values of $\gamma\neq 0.5$. Before the transition, all cars move freely and the average velocity is $\langle v_g\rangle=1$, while when the transition occurs, they are all stuck and $\langle v_g\rangle=0$. In the jamming state, the system reaches the asymptotic state formed by a separate rows of right and up cars along the diagonals from the upper-left to the lower-right corners, and this situation prevents the cars to move. As the system size increases, the critical density ρ_c tends to decrease giving rise to sharper jamming transition, and stabilize for high system sizes ($L\geqslant 300$).

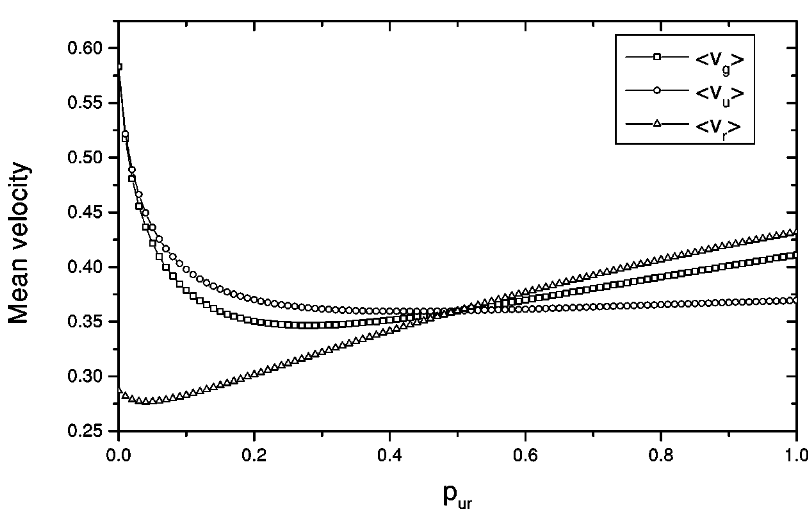

FIG. 3. The variation, in the periodic boundaries case, of the global, right, and up velocities as a function of p_{ur} for $p_{ru}=0.5$, $\rho=0.7$, and $L=100$.

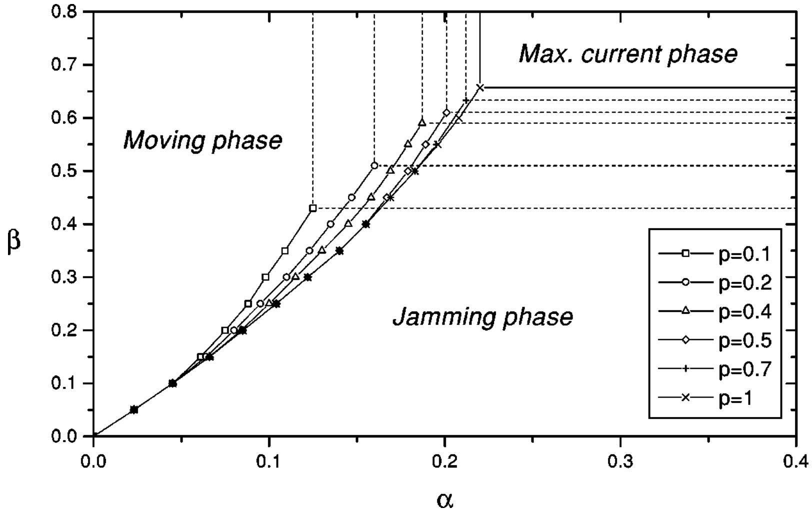

FIG. 4. Phase diagram in the (α,β) plane for several values of p; solid (dashed) lines corresponds to first-order (second-order) transition.

In the isotropic case, i.e., $p=p_{ur}=p_{ru}$ $(p\neq 0)$, Fig. 2 shows the variation of $\langle v_g\rangle=\langle v_r\rangle=\langle v_u\rangle$ as a function of the density for different values of p. In the $p=1$ case, the system undergoes a continuous transition from the freely moving phase to the jamming phase at $\rho=0.34$. Indeed, for low density $(\rho\leqslant 0.34)$ the mean velocity is equal to 1 (freely moving phase), while for $\rho\geqslant 0.34$, the mean velocity decreases (jamming phase) almost linearly with increasing density. For $0<p<1$, such transitions disappear (one cannot distinguish between freely moving and jamming phases) contrary to the result of Ref. [21] in which the continuous transition exists only at $\gamma=0.5$, otherwise the sharp jamming transition occurs. This disagreement means that the model we study is completely different than the one studied by Cuesta *et al.* [21]. Indeed, in our model the direction of the motion of cars at each time step depends on its direction of motion at the previous time; such a condition has not been considered in Ref. [21]. In order to understand the variation of the velocities versus the anisotropy for fixed density, one has to investigate the behavior of the mean velocities versus p_{ur} and p_{ru}. For the symmetry of the system, it suffices to study the variations $\langle v_r\rangle$ and $\langle v_u\rangle$ as a function of p_{ur} instead of studying them versus both parameters. However, the variation of the global, right, and up velocities as a function of p_{ur} for $p_{ru}=0.5$ and $\rho=0.7$ is given in Fig. 3. It is shown that, for $p_{ur}<p_{ru}$ $(p_{ur}>p_{ru})$, the number of up-moving (right-moving) cars is more important than the number of right-moving (up-moving) cars, therefore the value of the mean global velocity tends to the right (up) velocity. For $p_{ur}=p_{ru}$, the velocities are equal.

B. Open boundaries

In this case, the system exhibits three phases; moving, jamming, and maximal current phases [22–26]. These phases are governed by three factors: the flow of entering of cars, the flow of exiting the system, and the velocity of cars inside the network. In the isotropic case $(p=p_{ur}=p_{ru})$, the velocities increase with p, so the maximal current phase region is contracted to high values of the injecting rate α and the extracting rate β (Fig. 4). In the moving phase, for sufficiently small values of α the density ρ does not depend upon the values of p, while at higher values of α the density increases with decreasing p (Fig. 5), and contrary to the case of the jamming phase, the density increases with increasing value of p. In fact, for small values of p, the formation of

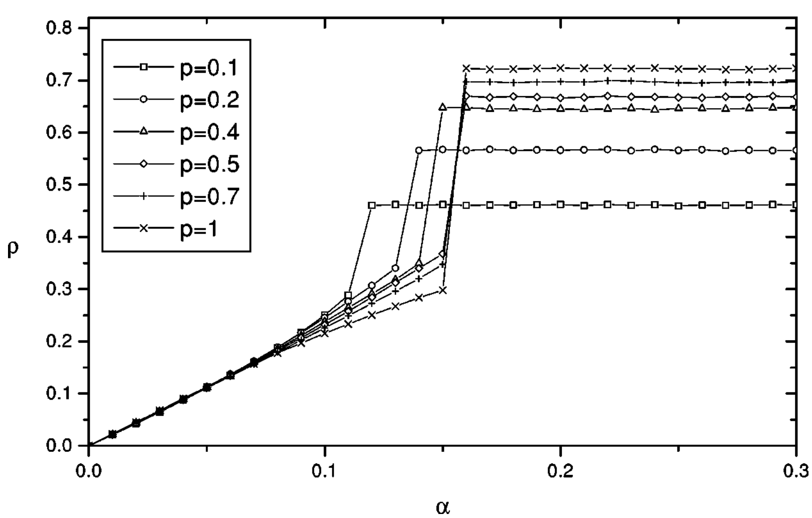

FIG. 5. The variation of the density ρ as a function of α for different values of p with $\beta=0.4$ and $L=101$.

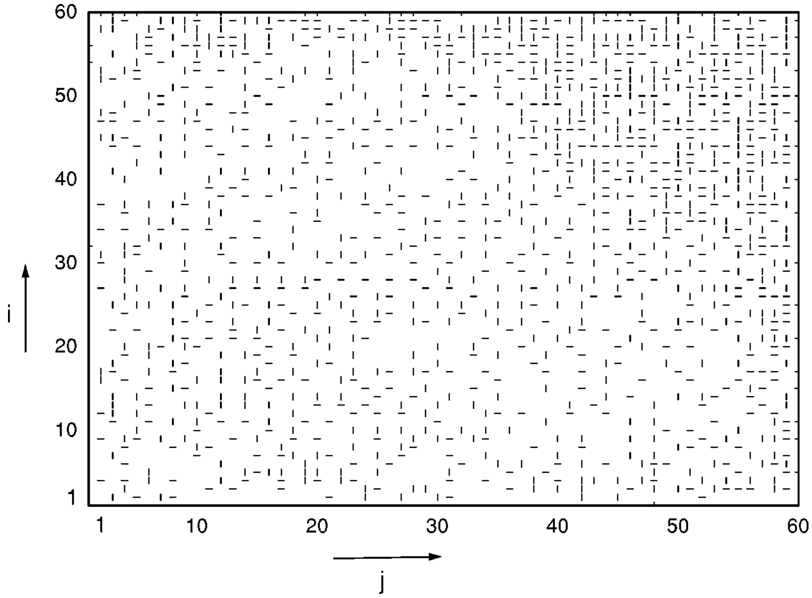

FIG. 6. Schematic configuration of the system in the isotropic case for $\beta=0.4$, $\alpha=0.148$, and $p=0.5$. The up-moving cars are indicated by the vertical bar and the right-moving cars by the horizontal bar; $L=60$.

small traffic-jam clusters blocks the cars inside the lattice, while at very low density such clusters disappear and the cars move freely even at very low values of p. Like in the asymmetric exclusion model (ASEP) [22–26], at the moving phase close to the jamming transition, the system is divided into two regions, namely, the high density region at the exit of the system, in our case this region is located at the vicinity of the upper-right exit (Fig. 6), and the low density region elsewhere. For low values of p, the system is blocked rapidly with the formation of local traffic-jam clusters, hence the density inside the high density region decreases. However, with increasing p, the traffic-jam clusters are unblocked and the empty sites located between these clusters become occupied, and then the density increases inside this region. Therefore, the area (density) of the high density region increases (decreases) with decreasing p. Like in the ASEP model, the transition between the moving and jamming phases occurs when the high density region invades the lattice. The transition between the jamming and the moving phases is a first-order transition (the density is discontinuous at the transition), while the transition between the moving phase (jamming phase) and the maximal current phase is of second order (the density is continuous at the transition), except in the case $p=1$, for which the transitions are of first order. However, the first-order transition occurs at $\alpha<\beta$, which means that the system reaches its jamming state at low densities because of the mutual blockage of cars moving in different directions. Moreover, the line transition is curved. Such a curvature is related to the dimension effect. Such a result is also obtained in the one-dimensional NS model with open boundaries but in the case $v_{\max}>1$ [19]. In fact, in both the cases the car has several possibilities of move. Indeed, in the NS model it has $v_{\max}$ choices, while in our case the car could move either right or upwards. In contrast with the transition from moving to jamming phases, the transition between the moving phase and the maximal current phase arises without the formation of a high density region in the exit of the system, while the density inside the lattice increases monotonically. In the case $p=1$, and in the absence of the clusters in the lattice, the increase of the density is not relevant at the vicinity of the transition, and since the density in the maximal current phase is sufficiently large because of the great number of occupied sites, the intersection between these two situations at the transition reflects an unstable equilibrium state which leads to a first-order transition.

The study of the anisotropic case, i.e., $p_{ur}\neq p_{ru}$, is summarized in Fig. 7(a) for $p_{ru}=0.1$ and Fig. 7(b) for $p_{ru}=1$. It is found that the topology of the phase diagram in the (α,β) plane is similar to the one obtained in the isotropic case (Fig. 5) (i.e., the system presents three different phases: moving, jamming, and maximal current phases. However, the behavior of the density as a function of β (α) shows that the first-order transitions between jamming phase (moving phase) and maximal current appear only for $p_{ru}=p_{ur}=p=1$ [Fig. 7(b)], otherwise these transitions are of second order. In the case $p_{ur}>p_{ru}$ $(p_{ur}<p_{ru})$, the up-moving (right-moving) cars entering on the bottom (left) of the lattice change their direction at the vicinity of the entry to become right-moving (up-moving) cars. Hence the move of cars is carried out at the vicinity of the bottom (left) entry in the right (up) direction as well as p_{ur} (p_{ru}) is larger than p_{ru} (p_{ur}). This augmentation of the density at the entrance prevents other cars to enter, which leads to an important diminution of the density inside the lattice at the moving phase. Then the transition occurs at a high critical value α_c. It is clear that the region of the maximal current phase in the (α,β) phase diagram [Figs. 7(a,b)] shrinks with increasing anisotropy (i.e., $|p_{ru}-p_{ur}|$). In fact, the increase of the anisotropy leads to the formation of a condensate band in one entering side which prevents the cars to enter the lattice. So, we have to increase α (for a fixed value of β) to overcome the traffic jam in the entrance, and then the transition line between the moving phase and the maximal current is shifted to the high value of α; while at the jamming phase, one must increase β (for a fixed value of α) in order to increase the traffic flow in the lattice, before reaching the maximal current phase. Then the line transition between jamming and

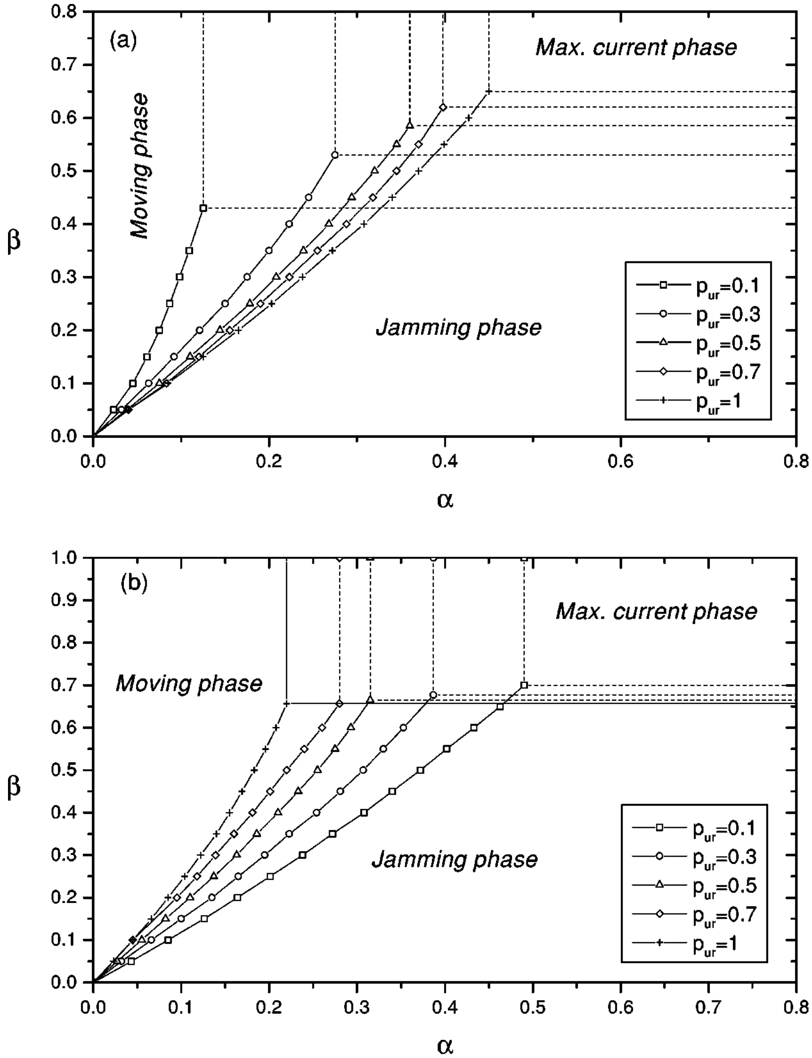

FIG. 7. Phase diagram in the (α,β) plane for several values of p_{ur}; (a) $p_{ru}=0.1$, (b) $p_{ru}=1$. Dashed lines correspond to second-order transition, while solid lines correspond to first order transition.

maximal current phases is shifted to a higher value of β.

Moreover, the density profile in the horizontal $(i=L/2,j=1,L)$ and vertical $(j=L/2,i=1,L)$ middle lines or in the oblique line from the left-bottom corner to right-upper corner decays in the maximal current phase with an exponent $\gamma \approx 0.25$ (Fig. 8), both in the anisotropic and isotropic cases. This means that the model belongs to another universality class than the models studied in one dimension, such as the ASEP $(\gamma=\frac{1}{2})$ [22,23] and the NS model for $v_{\max}>1$ $(\gamma \approx \frac{2}{3})$ [19].

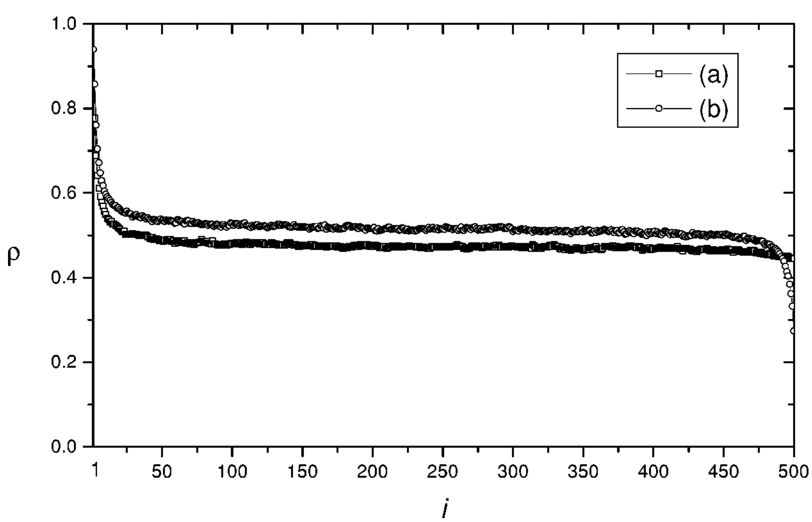

FIG. 8. The variation of the density profile, (a) $\rho(L/2,i)$ versus i (the horizontal middle line) and (b) $\rho(i,i)$ versus i (the oblique line) at the maximal current phase for $L=500$.

IV. CONCLUSION

In this paper, we have studied, in a version of the Biham-Middleton-Levine model of two-dimensional traffic flow with random sequential update, the anisotropy effect of the change of the directions of move, in the periodic and open boundaries conditions, in which the direction of motion of cars at each time step depends on its direction of motion at the previous time step; such a condition has not been considered in the model studied, in the periodic boundaries, by Cuesta *et al.* [21]. We have shown, in the periodic boundaries conditions, that the sharp jamming transition disappears when the cars could change their direction of motion at every time step (i.e., the sharp jamming transition exists only in the case $p=p_{ur}=p_{ru}=0$. In the open boundaries case, the system exhibits three phases, namely, moving phase, jamming phase, and maximal current phase. However, we have shown that the first-order line transition between the moving and the jamming phases is curved which takes place at $\alpha<\beta$. Furthermore, we have shown that the increase of the anisotropy disfavors the maximal current and the jamming phases. Moreover, in the anisotropic case, the transition between the jamming phase (or moving phase) and the maximal current phase is of second order while in the isotropic case, and when each car changes its direction of motion at every time step $(p_{ru}=p_{ur}=1)$, the transition is of first order. Finally, we have shown that the density profile in the horizontal (or vertical) middle line decays in the maximal current phase with an exponent $\gamma\approx\frac{1}{4}$, both in the anisotropic and isotropic cases.

[1] K. Nagel and M. Schreckenberg, J. Phys. I **2**, 2221 (1992).
[2] A. Schadschneider and M. Schreckenberg, J. Phys. A **26**, L679 (1993).
[3] L.C.Q. Vilar and A.M.C. de Souza, Physica A **211**, 84 (1994).
[4] K. Nagel and M. Paczuski, Phys. Rev. E **51**, 2909 (1995).
[5] M. Schreckenberg, A. Schadschneider, K. Nagel, and N. Ito, Phys. Rev. E **51**, 2939 (1995).
[6] J. Esser and M. Schreckenberg, Int. J. Mod. Phys. C **8**, 1025 (1997).
[7] P.M. Simon and K. Nagel, Phys. Rev. E **58**, 1286 (1998).
[8] T. Nagatani, J. Phys. A **26**, L1015 (1993).
[9] S.C. Benjamin, N.F. Johnson, and P.M. Hui, J. Phys. A **29**, 3119 (1996).
[10] T. Nagatani, Physica A **233**, 137 (1996).
[11] T. Nagatani, J. Phys. Soc. Jpn. **62**, 2656 (1993).
[12] K.H. Chung, P.M. Hui, and G.Q. Gu, Phys. Rev. E **51**, 772 (1995).
[13] T. Nagatani, Phys. Rev. E **58**, 4271 (1998).
[14] T. Komatsu and S. Sasa, Phys. Rev. E **52**, 5574 (1995).
[15] O. Biham, A.A. Middleton, and D.A. Levine, Phys. Rev. A **46**, R6124 (1992).
[16] T. Nagatani, J. Phys. Soc. Jpn. **62**, 2656 (1993).
[17] S. Tadaki and M. Kikuchi, Phys. Rev. E **50**, 4564 (1994).
[18] T. Horiguchi and T. Sakakibara, Physica A **252**, 388 (1998).
[19] S. Cheybani, J. Kertész, and M. Schreckenberg, Phys. Rev. E **63**, 016108 (2001).
[20] A. Benyoussef, N. Boccara, H. Chakib, and H. Ez-Zahraouy, Chin. J. Phys. (Taipei) **39**, 428 (2001).
[21] José A. Cuesta, Froilán C. Martínez, Juan M. Molera, and Angel Sánchez, Phys. Rev. E **48**, R4175 (1993); Froilán C. Martínez, José A. Cuesta, and Juan M. Molera, *ibid.* **51**, R835 (1995); Juan M. Molera, Froilán C. Martínez, and José A. Cuesta, *ibid.* **51**, 175 (1995).
[22] B. Derrida, M.R. Evans, and D. Mukamel, J. Phys. A **26**, 4911 (1993).
[23] B. Derrida and M. R. Evans, in *Non Equilibrium Statistical Mechanics in One Dimension*, edited by V. Privman (Cambridge University Press, Cambridge, 1996).
[24] M.R. Evans, N. Rajewsky, and E.R. Speer, J. Stat. Phys. **95**, 45 (1999).
[25] J. de Grier and B. Nienhuis, Phys. Rev. E **59**, 4899 (1999).
[26] A. Benyoussef, H. Chakib, and H. Ez-Zahraouy, Eur. Phys. J. B **8**, 275 (1999).